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ANOTHER PROOF OF THE THEOREM CONCERNING ARTIFICIAL SINGULARITIES

BY MAXIME BÔCHER

THE theorem considered by Professor Curtiss in the preceeding note is so important that still another simple proof may not be out of place. The proof which I give is a simplification of the second proof given by Osgood in the *Bulletin of the American Mathematical Society* for June 1896, p. 301.

LEMMA. *If $f(z)$ is continuous throughout a region S, and is analytic at every point of this region except at a point c which is not on the boundary of S, then f is analytic at c also.*

Consider a circle K with c as centre and lying wholly in S. The integral of $f(z)$ taken around any regular closed path in K is zero. To prove this we consider three cases:

1) If this closed path does not surround or pass through c, $f(z)$ is analytic everywhere upon and within the path. Hence the integral is zero.

2) If the path surrounds c, the integral around it is equal to the integral around an arbitrarily small circle having c as centre. Accordingly its absolute value cannot exceed $2\pi rM$ where r is the radius of this small circle and M is the maximum of $|f(z)|$ on its circumference. But since as r approaches zero, $\lim M = |f(c)|$, we have $\lim 2\pi rM = 0$. Hence the value of our integral must be zero.

3) If the path passes through c, the integral around it is equal to the integral along an arbitrarily short arc of length l of this path having c as its middle point plus the integral along a path connecting the extremities of this arc and of length less than $2l$. Accordingly the absolute value of this integral cannot exceed $3lM$, where M is the maximum of $|f(z)|$ on the path just described. Since this vanishes with l, our integral is again zero.

Since $f(z)$ is continuous throughout K and its integral around any regular closed path in K is zero, it follows from Morera's theorem[*] that $f(z)$ is analytic at c as well as elsewhere.

THEOREM. *If $f(z)$ is analytic throughout a region S except at a point c which is not on the boundary of S, and if $f(z)$ remains finite in the neighborhood of c, then $f(z)$ approaches a finite limit as z approaches c, and if this limiting value be taken as the definition of $f(c)$, then f is analytic at c.*

For the function $(z-c)f(z)$ is by our lemma analytic at c if we define it as having the value zero there. Hence we may write

$$(z-c)f(z) = a_1(z-c) + a_2(z-c)^2 + \cdots,$$

and therefore, when $z \neq c$,

$$f(z) = a_1 + a_2(z-c) + \cdots.$$

Hence as z approaches c, $\lim f(z) = a_1$, and if we define $f(c) = a_1$, f is analytic at c.

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[*] Cf. Osgood, *Funktionentheorie*, vol. 1, p. 256. This theorem is so easily proved and so useful that it may well be introduced at an early stage even in an elementary course on the theory of functions.